The Eigenvalues and Eigenfunctions of $Lu := -u''$

Let $Lu := -u''$ where $u(x)$ is in the space $C_0^2[a,b]$ of twice continuously differentiable real-valued functions on the interval $a \le x \le b$ with the *boundary conditions* $u(a) = 0$ and $u(b) = 0$ (the subscript 0 in $C_0^2[a,b]$ is to remind us of these boundary conditions). This differential operator, $L = -D^2$, with these boundary conditions, arises in many applications. We inserted the minus sign in the definition $Lu := -u''$ since this way it will turn out (see below) that the eigenvalues of L are positive.

Example. The oscilations $w(x,t)$ of a guitar string on the interval $a \le x \le b$ and time $t \ge 0$. Then w satisfies the *wave equation*

$$\frac{\partial^2 w(x,t)}{\partial t^2} = c^2 \frac{\partial^2 w(x,t)}{\partial x^2}, \tag{1}$$

where c is a constant depending on the density and tension of the string. Since the ends of the guitar string are fixed, w satisfies the boundary conditions $w(a,t) = w(b,t) = 0$.
Oscilations of the special form $w(x,t) = u(x)\phi(t)$ are called *standing waves.* Plugging this into (1), we get

$$u(x)\ddot{\phi}(t) = c^2 u''(x)\phi(t).$$

Now we separate the variables, putting the functions of t on one side and the functions of x on the other, giving

$$\frac{\ddot{\phi}(t)}{c^2\phi(t)} = \frac{u''(x)}{u(x)}.$$

Because the left side does not depend on x and the right side on t, both sides are some constant $-\lambda$; we added the minus sign since this way it will turn out (see below) that $\lambda > 0$. Consequently

$$-u''(x) = \lambda u(x) \qquad \text{and} \qquad -\ddot{\phi}(t) = c^2\lambda\phi(t).$$

Also, the boundary conditions on $w(x,t)$ imply that $u(a) = 0$ and $u(b) = 0$. Thus, $u(x)$ is an eigenfunction of $Lu := -u''$ and λ the corresponding eigenvalue. It is the differential equation for $u(x)$ along with the boundary condition that will determine the eigenvalues λ. Only then does one solve the (simple) differential equation for $\phi(t)$.

We will use the inner product

$$\langle u,\, v\rangle = \int_a^b u(x)v(x)\,dx.$$

Compute the Adjoint L^* of L.
By definition, the adjoint, L^* has the property $\langle Lu,\, v\rangle = \langle u,\, L^*v\rangle$ for all functions u, v in $C_0^2[a,b]$, that is,

$$\int_a^b -u''(x)v(x)\,dx = \int_a^b u(x)L^*v(x)\,dx,$$

so we want to get the derivatives off of u on the left-hand side. Integrating by parts *twice* does the job. Moreover, because of the boundary conditions on u and v, the boundary terms when integrating by parts are zero. This gives:

$$\int_a^b u''(x)v(x)\,dx = \int_a^b u(x)[-v(x)'']\,dx,$$

Comparing the last two equations, we find that $L^*v = -v''$, so in this case, $L^* = L$ so L – with these boundary conditions – is *self-adjoint*. Thus we know immediately that its eigenvalues λ are real numbers.

Find the Eigenvalues and Eigenfunctions of L.
With hindsight, for simplicity we will use the interval $0 \le x \le \pi$. We want to solve $Lu = \lambda u$, that is, $-u'' = \lambda u$, where u also satisfies the boundary conditions. The trivial solution $u(x) \equiv 0$ is never considered to be an eigenfunction.
First we claim that $\lambda > 0$ (we already know that λ is real). Since L with our boundary conditions is self-adjoint, this immediately tells us that all its eigenvalues are real. There are several ways to see thtat the eigenvalues are positive. One way is to try the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ separately. However the following approach works in many more problems: Take the inner product of both sided of $-u'' = \lambda u$ with u, so $\langle Lu,\, u\rangle = \lambda\langle u,\, u\rangle$, that is,

$$\int_0^\pi -u''(x)u(x)\,dx = \lambda\int_0^\pi u^2\,dx \tag{2}$$

and integrate the left-hand side by parts. Because of the boundary conditions, the boundary terms in the integration by parts are zero. This gives

$$\langle Lu,\, u\rangle = \int_0^\pi -u''(x)u(x)\,dx = \int_0^\pi u'(x)u'(x)\,dx.$$

Using this in the left-hand side of (2) we find that

$$\int_0^\pi u'^2\,dx = \lambda\int_0^\pi u^2\,dx. \tag{3}$$

If $\lambda = 0$ in (3), this implies that $u'(x) \equiv 0$. Thus $u(x) \equiv$constant. Because $u(0) = 0$, this constant is zero so in this case u is not an eigenfunction. Because $u(x) \not\equiv 0$, this gives the useful formula

$$\lambda = \frac{\int_0^\pi u'^2\,dx}{\int_0^\pi u^2\,dx} > 0.$$

so $\lambda > 0$ and we will write $\lambda = k^2 > 0$
Then the equation $Lu = \lambda u$ is $u'' + k^2u = 0$ whose general solution is $u(x) = A\cos kx + B\sin kx$. Now we use the boundary conditions. The condition $u(0) = 0$ shows that $A = 0$ while the condition $u(\pi) = 0$ means $0 = B\sin k\pi$. If we let $B = 0$, then $u(x) \equiv 0$, which is not allowed. Thus we need $\sin k\pi = 0$. This is only satisfied if k is an *integer*, $k = 1,\, 2,\, 3,\, \ldots$.

SUMMARY The eigenfunctions of $Lu := -u''$ are $u_k(x) = \sin kx$, $k = 1, 2, 3, \ldots$ with c0rresponding eigenvalues $\lambda = k^2$. Since the u_k are eigenfunctions of a self-adjoint operator with distinct eigenvalues, by general theory, they are automatically orthogonal in this inner product.

[Last revised: May 5, 2013]