Final Exam

Math 312, Fall 2012, Prof. Jauregui

You have 120 minutes.

No books, phones, calculators, talking, etc. You are allowed your own hand-written 3×5 inch note card, both sides, but **no other materials**. You may write on the back side of the sheets if necessary.

PLEASE WRITE LEGIBLY AND SHOW YOUR WORK.
YOU MAY NOT RECEIVE FULL CREDIT IF YOU DO NOT SHOW WORK.

Good luck!

1. *(6 points)* Suppose $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear transformation represented by a matrix, A.

   (a) What possible values could the rank of A be? Why?

   (b) What possible values could the dimension of the kernel of A be? Why?

   (c) Suppose the rank of A is as large as possible. What is the dimension of $\ker(A)^{\perp}$? Explain.

2. *(8 points)* True or false? **If false, give a reason.**

   (a) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a collection of vectors in $\mathbb{R}^5$, then the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ must be a three-dimensional subspace of $\mathbb{R}^5$.

   (b) The set of polynomials in $\mathbb{P}_4$ satisfying $p(0) = 2$ is a subspace of $\mathbb{P}_4$.

   (c) If $\vec{x}^*$ is a least-squares solution to $A\vec{x} = \vec{b}$, then $A\vec{x}^*$ is orthogonal to the image of A.

(d) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are vectors in $\mathbb{R}^3$ such that none is a scalar multiple of either of the others, then these vectors are linearly independent.

3. *(9 points)* Let $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{bmatrix}$$

(a) Find a basis for the kernel of A.

(b) Find a basis for the image of A.

(c) Suppose B is some invertible 3×3 matrix. Is it possible for $L(\vec{x}) = AB\vec{x}$ to be an isomorphism (where A is defined as above)? Explain carefully.

4. *(6 points)* Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Is the origin a stable equilibrium of the discrete dynamical system $\vec{x}_{k+1} = A\vec{x}_k$? Explain.

5. *(10 points)* Consider the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}.$$

(a) If $\vec{x} \in \mathbb{R}^2$ is a unit vector, what is the largest that $\|A\vec{x}\|$ could possibly be?

(b) Compute an SVD of A.

(c) Using part (b), compute a rank-1 approximation to A.

6. *(10 points)* Consider the vector space $\mathbb{P}_2$ with basis $\mathcal{B} = \{x^2, x, 1\}$. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation that sends a polynomial $p(x)$ to $\frac{d}{dx}\left((x-1)p(x)\right)$. (That is, T first multiplies $p(x)$ by $(x-1)$, then takes the derivative of the result.)

(a) Find the matrix representation of T with respect to the basis $\mathcal{B}$.

(b) What are the eigenvalues of T?

(c) Find an eigenbasis $\mathcal{C}$ of T. (Be sure to express your answer as elements of $\mathbb{P}_2$, not vectors in $\mathbb{R}^3$.)

7. *(6 points)* Let A be an $m \times n$ matrix, and suppose $\vec{v}$ and $\vec{w}$ are orthogonal eigenvectors of $A^T A$. Prove that $A\vec{v}$ and $A\vec{w}$ are orthogonal.

8. *(6 points)* Of the following three matrices, one can be orthogonally diagonalized; one can be diagonalized (but not orthogonally); and one cannot be diagonalized at all. Identify these, **fully explaining your reasoning.**

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

9. *(10 points)* Let V be the vector space spanned by the two functions e^x and e^{-x}, considered only on the interval $[-1, 1]$. Give V the L^2 inner product:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

(a) Prove that $\mathcal{B} = \{e^x + e^{-x}, e^x - e^{-x}\}$ forms an orthogonal basis of V.

(b) Find the best approximation in V (with respect to the L^2 inner product) of the function $g(x) = x$. (Hint: think orthogonal projection. **Leave your answer in terms of integrals that could be evaluated easily on a computer.**)

(c) Let $\mathcal{C}$ be the basis $\{e^x, e^{-x}\}$ of V. Find the change-of-basis matrices $P_{\mathcal{B}\to\mathcal{C}}$ and $P_{\mathcal{C}\to\mathcal{B}}$, clearly indicating which is which.

10. *(5 points)* Answer the following short questions, **fully explaining your steps**.

(a) If A is orthogonally diagonalizable, must A be symmetric? Why or why not?

(b) Suppose P is a matrix satisfying $P^3 = P$. What are the possible eigenvalues of P?