Signature PRINTED NAME

My signature above certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

Math 312 **Final Exam** Jerry L. Kazdan
May 1, 2013 12:00 – 2:00

DIRECTIONS This exam has two parts. Part A has shorter 5 questions, (8 points each so total 40 points) while Part B has 6 problems (15 points each, so total is 90 points). Maximum score is thus 130 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (8 points each, so 40 points).

Score	
A-1	
A-2	
A-3	
A-4	
A-5	
B-1	
B-2	
B-3	
B-4	
B-5	
B-6	
Total	

A–1. Suppose $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear transformation represented by a matrix, A.

a) What possible values could the rank of A be? Why?

b) What possible values could the dimension of the kernel of A be? Why?

c) Suppose the rank of A is as large as possible. What is the dimension of $\ker(A)^\perp$? Explain.

A–2. Let $\vec{v}$ be an eigenvector of an invertible matrix A. Which of the following are necessarily true? Please give your reasoning.

I. $\vec{v}$ is an eigenvector of A^{-1}. II. $\vec{v}$ is an eigenvector of A^2. III. $\vec{v}$ is an eigenvector of $A+I$.

A–3. True or false? **If false, give a reason.**

a) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a collection of non-zero vectors in $\mathbb{R}^5$, then the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ must be a three-dimensional subspace of $\mathbb{R}^5$.

b) The set of polynomials in $\mathcal{P}_4$ satisfying $p(0) = 2$ is a subspace of $\mathcal{P}_4$.

c) If $\vec{x}$ is a least-squares solution to $A\vec{x} = \vec{b}$, then $A\vec{x}$ is orthogonal to the image of A.

d) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthonormal vectors in $\mathbb{R}^3$, then these vectors are linearly independent.

e) If the matrix A is both invertible and diagonalizable, then A^{-1} is diagonalizable.

A–4. Consider the matrix $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$. If $\vec{x} \in \mathbb{R}^2$ is a unit vector, what is the largest that $\|A\vec{x}\|$ could possibly be?

A–5. Let A be an $m \times n$ matrix, and suppose $\vec{v}$ and $\vec{w}$ are orthogonal eigenvectors of $A^T A$. Show that $A\vec{v}$ and $A\vec{w}$ are orthogonal.

PART B Six questions, 15 points each (so 90 points total).

B–1. Find an orthogonal matrix R that diagonalizes $A := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

B–2. Let V be the vector space spanned by the two functions e^x and e^{-x}, considered only on the interval $[-1, 1]$. Give V the L^2 inner product:

$$\langle f,\, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

a) Prove that $\mathcal{B} = \{e^x + e^{-x},\, e^x - e^{-x}\}$ forms an orthogonal basis of V.

b) Find the best approximation in V (with respect to the L^2 inner product) of the function $g(x) = x$. [HINT: think orthogonal projection. **Leave your answer in terms of integrals that could be evaluated easily using a computer program.**]

B–3. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs.

One has data on which location the cars are returned daily:

- RENTED AT AIRPORT: 2% are returned to the City and 25% to the Suburbs. The rest are returned to the Airport.
- RENTED IN CITY : 10% returned to Airport, 10% returned to Suburbs. The rest are returned to the City.
- RENTED IN SUBURBS: 25% are returned to the Airport and 2% to the city. The rest are returned to the Suburbs.

If initially there are 35 cars at the Airport, 150 in the city, and 35 in the suburbs, what is the long-term distribution of the cars?

B–4. Say $\vec{x}(t)$ is a solution of $\dfrac{d\vec{x}}{dt} = A\vec{x}$, where $A := \begin{pmatrix} c & 5 \\ 5 & c \end{pmatrix}$. For which value(s) of the real constant c does *every* solution $\vec{x}(t)$ tends to zero as $t \to \infty$?

B–5. Of the following three matrices, which (if any) can be orthogonally diagonalized; which can be diagonalized (but not orthogonally); and which cannot be diagonalized at all. Identify these, **fully explaining your reasoning.**

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 7 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

B–6. Let $A := \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$. By a routine computation the matrix $A^*A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ has eigenvalues 3 and 1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a) Use this to find the singular value decomposition of A.

b) Find the best rank 1 approximation to A.