Math 312 **Final Exam** Jerry L. Kazdan
May 1, 2013 12:00 – 2:00

DIRECTIONS This exam has two parts. Part A has shorter 5 questions, (8 points each so total 40 points) while Part B has 6 problems (15 points each, so total is 90 points). Maximum score is thus 130 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (8 points each, so 40 points).

A–1. Suppose $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear transformation represented by a matrix, A.

a) What possible values could the rank of A be? Why?

SOLUTION: $0 \le \operatorname{rank}(A) \le 4$. For instance, if A is the zero matrix, its rank is 0.

b) What possible values could the dimension of the kernel of A be? Why?

SOLUTION: $2 \le \dim\ker(A) \le 6$. For instance, if A is the zero matrix, the dimension of its kernel is 6. By the rank-nullity theorem, the dimension of its kernel is at least 2.

c) Suppose the rank of A is as large as possible. What is the dimension of $\ker(A)^\perp$? Explain.

SOLUTION: If $\operatorname{rank}(A) = 4$, then as above, $\dim\ker(A) = 2$ so the dimension of its orthogonal complement is 4.

A–2. Let $\vec{v}$ be an eigenvector of an invertible matrix A. Which of the following are necessarily true? Please give your reasoning.

I. $\vec{v}$ is an eigenvector of A^{-1}. II. $\vec{v}$ is an eigenvector of A^2. III. $\vec{v}$ is an eigenvector of $A+I$.

SOLUTION: All of these are TRUE.

I. If $A\vec{v} = \lambda\vec{v}$, then $\vec{v} = \lambda A^{-1}\vec{v}$ so $A^{-1}\vec{v} = (1/\lambda)\vec{v}$. [Since A is invertible, then $\lambda = 0$ cannot be an eigenvalue].

II. $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda^2\vec{v}$.

More generally, for any integer k (positive, negative, or zero) $A^k\vec{v} = \lambda^k\vec{v}$.

III. $(A + I)\vec{v} = \lambda\vec{v} + \vec{v} = (\lambda + 1)\vec{v}$.

A–3. True or false? **If false, give a reason.**

a) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a collection of non-zero vectors in $\mathbb{R}^5$, then the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ must be a three-dimensional subspace of $\mathbb{R}^5$.

SOLUTION: FALSE. The span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a subspace of dimension *at most* 3, but if the $\vec{v}_j$ are linearly dependent, its dimension could be 1 or 2 (but not 0).

b) The set of polynomials in $\mathcal{P}_4$ satisfying $p(0) = 2$ is a subspace of $\mathcal{P}_4$.

SOLUTION: FALSE. It is not a subspace because it does not contain the 0 polynomial.

c) If $\vec{x}$ is a least-squares solution to $A\vec{x} = \vec{b}$ then $A\vec{x}$ is orthogonal to the image of A.

SOLUTION: FALSE. $A\vec{x}$ is *in* the image of A. It could be orthogonal to the image of A only in the special unlikely case where $A\vec{x} = 0$.

d) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthonormal vectors in $\mathbb{R}^3$, then these vectors are linearly independent.

SOLUTION: TRUE. Any orthogonal non-zero vectors are linearly independent since if

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = 0,$$

then taking the inner product of this with $\vec{v}_1$ we find that $a\langle\vec{v}_1, \vec{v}_1\rangle = 0$. Thus $a = 0$. Similarly $b = 0$ and $c = 0$.

e) If the matrix A is both invertible and diagonalizable, then A^{-1} is diagonalizable.

SOLUTION: TRUE. If A diagonalizable, then $A = SDS^{-1}$, where D is a diagonal matrix. Now take the inverse of both sides.

A–4. Consider the matrix $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$. If $\vec{x} \in \mathbb{R}^2$ is a unit vector, what is the largest that $\|A\vec{x}\|$ could possibly be?

SOLUTION: The key fact is that if M is a self-adjoint $n\times n$ with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ (necessarily real) then for any $\vec{x}$ we have

$$\lambda_1\|\vec{x}\|^2 \le \langle\vec{x},\, M\vec{x}\rangle \le \lambda_n\|\vec{x}\|^2.$$

Since

$$\|A\vec{x}\|^2 = \langle A\vec{x},\, A\vec{x}\rangle = \langle\vec{x},\, A^*A\vec{x}\rangle,$$

the matrix $M := A^*A$ is self adjoint and positive semi-definite, the above expression is largest if $\vec{x}$ is an eigenvector of A^*A corresponding to its largest eigenvalue, $\lambda_{\max}$. Thus

$$\|A\vec{x}\|^2 \le \lambda_{\max}\|\vec{x}\|^2.$$

Because the problem specifies that $\vec{x}$ is a unit vector, then

$$\|A\vec{x}\|^2 \le \lambda_{\max}.$$

We now compute $\lambda_{\max}$. The computation is routine. Note that we use the matrix $M = A^*A$, *not* A.

$$A^*A = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}.$$

Then the characteristic polynomial is

$$\det(A^*A - \lambda I) = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4).$$

Consequently $\lambda_{\max} = 9$ so $\|A\vec{x}\| \le 3$.

A–5. Let A be an $m \times n$ matrix, and suppose $\vec{v}$ and $\vec{w}$ are orthogonal eigenvectors of $A^T A$. Show that $A\vec{v}$ and $A\vec{w}$ are orthogonal.

SOLUTION: Say $A^T A\vec{w} = \lambda\vec{w}$. Then

$$\langle A\vec{v},\, A\vec{w}\rangle = \langle \vec{v},\, A^T A\vec{w}\rangle = \langle \vec{v},\, \lambda\vec{w}\rangle = \lambda\langle \vec{v},\, \vec{w}\rangle.$$

But $\vec{v}$ and $\vec{w}$ are given to be orthogonal so the result follows. Note that in this we never needed to use that $\vec{v}$ is also an eigenvector of $A^T A$

PART B Six questions, 15 points each (so 90 points total).

B–1. Find an orthogonal matrix R that diagonalizes $A := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

SOLUTION: The characteristic polynomial is

$$\det(A - \lambda I) = (2-\lambda)\det\begin{pmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} = (2-\lambda)[(1-\lambda)^2 - 1] = -\lambda(\lambda-2)^2.$$

Thus the eigenvalues are $\lambda_1 = 0,\ \lambda_2 = \lambda_3 = 2$ For λ_1 we find $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = \lambda_3 = 2$ we want the kernel of the matrix

$$A - 2I = \begin{pmatrix} -1 & -1 & 0 \\ =1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that this kernel consists of all vectors of the form $\begin{pmatrix} a \\ -a \\ c \end{pmatrix}$. One simple orthogonal basis is $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The orthogonal matrix R that diagonalizes A has *orthonormal* eigenvectors as its columns. The orthogonal eigenvectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ do the job simply by making them into unit vectors. Thus

$$R = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

B–2. Let V be the vector space spanned by the two functions e^x and e^{-x}, considered only on the interval $[-1, 1]$. Give V the L^2 inner product:

$$\langle f,\, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

a) Prove that $\mathcal{B} = \{e^x + e^{-x},\, e^x - e^{-x}\}$ forms an orthogonal basis of V.

SOLUTION Write $E_1(x) := e^x + e^{-x}$ and $E_2(x) := e^x - e^{-x}$. Note that E_1 is an even function and E_2 is an odd function. In this inner product, an even function and an odd function are always orthogonal.

To show they are a basis, we need to show they are both linearly independent and span. One way to see that they are linearly independent is to use that they are orthogonal (see Problem A-3d above). There are many other approaches.

They span V since

$$e^x = \frac{E_1 + E_2}{2} \qquad \text{and} \qquad e^{-x} = \frac{E_1 - E_2}{2}.$$

Thus any linear combination of e^x and e^{-x} can also be written as a linear combination of E_1 and E_2.

b) Find the best approximation in V (with respect to the L^2 inner product) of the function $g(x) = x$. [HINT: think orthogonal projection. **Leave your answer in terms of integrals that could be evaluated easily using a computer program.**]

SOLUTION: To find the best approximation of $g(x) = x$ in V, we use E_1 and E_2 as an orthogonal basis for V and want to write

$$x = aE_1 + bE_2 + h(x), \tag{1}$$

where a and b are constants and $h(x)$ is orthogonal to both E_1 and E_2. Then $aE_1 + bE_2$ is the orthogonal projection of x into V and is the desired best approximation of x in V:

$$\mathrm{Proj}_V(x) = aE_1(x) + bE_2(x).$$

Our task is to find a and b. To compute a take the inner product of both sides of equation (1) with E_1 and use that E_1 is orthogonal to both E_2 and h. Thus

$$\langle x,\, E_1\rangle = a\|E_1\|^2 \qquad \text{so} \qquad a = \frac{\langle x,\, E_1\rangle}{\|E_1\|^2}.$$

Similarly

$$\langle x,\, E_2\rangle = b\|E_2\|^2 \qquad \text{so} \qquad b = \frac{\langle x,\, E_2\rangle}{\|E_2\|^2}.$$

These formulas for a and b just involve the inner product and hence straightforward but tedious integrals. [One can save time by observing that $a = 0$ because the numerator is the inner product of an odd and an even function.]

B–3. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- RENTED AT AIRPORT: 2% are returned to the City and 25% to the Suburbs. The rest are returned to the Airport.

- Rented in City : 10% returned to Airport, 10% returned to Suburbs. The rest are returned to the City.
- Rented in Suburbs: 25% are returned to the Airport and 2% to the city. The rest are returned to the Suburbs.

If initially there are 35 cars at the Airport, 150 in the city, and 35 in the suburbs, what is the long-term distribution of the cars?

Solution: Let A_k, C_k, and S_k denote the number of cars at the Airport, City, and Suburbs, respectively, on day k. The above data tells us that on day $k+1$

$$\begin{aligned} A_{k+1} &= .73A_k + .10C_k + .25S_k \\ C_{k+1} &= .02A_k + .80C_k + .02S_k \\ S_{k+1} &= .25A_k + .10C_k + .73S_k \end{aligned} \qquad \text{and write} \qquad D_k = \begin{pmatrix} A_k \\ C_k \\ S_k \end{pmatrix} \qquad \text{so} \qquad D_0 = \begin{pmatrix} 35 \\ 150 \\ 35 \end{pmatrix}.$$

The vector D_k gives the distribution of the cars on day k. Note for every day the sum of the components of D_k is always 220 since the same cars just get moved around. The transition matrix for this Markov chain is

$$T := \begin{pmatrix} .73 & .10 & .25 \\ .02 & .80 & .02 \\ .25 & .10 & .73 \end{pmatrix}$$

The long-term distribution is $D := \lim_{k\to\infty} D_k$. From the theory, $D = TD$ so D is an eigenvector of T associated with the eigenvalue 1. Thus we solve the equations $(T-I)D = 0$.

First we seek an eigenvector $\vec{v}$ of $T\vec{v} = \vec{v}$ with no special normalization. This gives $\vec{v} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$.

Since the sum of the components of $\vec{v}$ is 11 and to find D the sum of the components should be 220, then $D = \frac{220}{11}\vec{v} = 20\vec{v} = \begin{pmatrix} 100 \\ 20 \\ 100 \end{pmatrix}$. It is often useful to compute the corresponding probability vector P the sum of whose components add to 1. Then $P = \frac{1}{11}\vec{v}$.

B–4. Say $\vec{x}(t)$ is a solution of $\dfrac{d\vec{x}}{dt} = A\vec{x}$, where $A := \begin{pmatrix} c & 5 \\ 5 & c \end{pmatrix}$. For which value(s) of the real constant c does *every* solution $\vec{x}(t)$ tends to zero as $t \to \infty$?

Solution: Since A is a symmetric matrix, it can be diagonalized (or one can compute that the eigenvalues are $c+5$ and $c-5$, which are distinct, hence the matrix is diagonalizable). Let λ_1 and λ_2 be the eigenvalues with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Then by using $\vec{v}_1$ and $\vec{v}_2$ as a basis for the plane, the general solution $\vec{x}(t)$ of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2, \tag{2}$$

where the coefficients c_1 and c_2 can be determined by the initial conditions. Since we want *all* solution to tend to 0 as $t \to \infty$, we need to know more about the eigenvalues. By a simple computation, they are $\lambda_1 = c+5$ and $\lambda_2 = c-5$. From the formula (2)we need *both* $c+5 < 0$ and $c-5 < 0$. Thus the restriction on c is $c < -5$.

B–5. Of the following three matrices, which (if any) can be orthogonally diagonalized; which can be diagonalized (but not orthogonally); and which cannot be diagonalized at all. Identify these **fully explaining your reasoning.**

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 7 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

SOLUTION:

A is a symmetric matrix and is thus orthogonally diagonalizable (by the Spectral Theorem).

B is *not* diagonalizable since all of its eigenvalues are 0 so that if it were diagonalizable, it would have to be the 0 matrix. Alternately, one can show that the geometric multiplicity of the eigenvalue 0 is 2 (not 3).

C is diagonalizable since its eigenvalues: 7, 1, and 2, are distinct. It is not orthogonally diagonalizable since the only such are symmetric matrices [Homework Set 9, #8].

B–6. Let $A := \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$. By a routine computation the matrix $A^*A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ has eigenvalues 3 and 1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

a) Use this to find the singular value decomposition of A.

SOLUTION: The singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{1} = 1$ with corresponding *orthonormal* eigenvectors $\vec{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. Note that it is traditional to label these so that σ_1 is the *largest* singular value. [If you number these differently, then you must use a consistent convention in part b) since the best rank 1 approximation is always associated with the largest singular value.]

Then the orthonormal $\vec{u}_j$'s are

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix},$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{1} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

The singular value decomposition of A is then

$$\begin{aligned} A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T =& \frac{\sqrt{3}}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \\ =& \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 2 & -2 \\ -1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 & -1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned} \tag{3}$$

Equivalently, $A = U\Sigma V^T$, where U is an orthogonal 3×3 matrix whose columns are $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ with $\vec{u}_3$ a unit vector orthogonal to $\vec{u}_1$ and $\vec{u}_2$ (we never need to compute $\vec{u}_3$ explicitly), V an orthogonal 2×2 matrix whose columns are $\vec{v}_1$ and $\vec{v}_2$, and Σ a 3×2 matrix containing the singular values of A. Thus

$$U = \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{2} & \vdots \\ 2/\sqrt{6} & 0 & \vec{u}_3 \\ -1/\sqrt{6} & 1/\sqrt{2} & \vdots \end{pmatrix}, \qquad \Sigma := \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad V := \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

b) Find the best rank 1 approximation to A.

SOLUTION: From equation (3), this is $\sigma_1 \vec{u}_1 \vec{v}_1^T = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 2 & -2 \\ -1 & 1 \end{pmatrix}$