Math 312 | Exam 1 | Jerry L. Kazdan
Feb. 12, 2013 | | 9:00 – 10:20

DIRECTIONS This exam has two parts. Part A has shorter 5 questions, (10 points each so total 50 points) while Part B had 4 problems (15 points each, so total is 60 points). Maximum score is thus 110 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (10 points each, so 50 points).

A–1. Which of the following sets are linear spaces? [If not, why not?]

a) In $\mathbb{R}^3$, the span of $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

b) The points $\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 5$.

c) The set of points $(x,\, y) \in \mathbb{R}^2$ with $y = 2x + x^2$.

d) The set of once differentiable solutions $u(x)$ of $u' + 3x^2 u = 0$. [You are *not* being asked to solve this equation.]

e) The set of polynomials $p(x)$ of degree at most 2 with $p'(1) = 0$.

A–2. Let $\mathcal{S}$ be the linear space of 2×2 matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $2a + d = 0$. Find a basis and compute the dimension of $\mathcal{S}$.

A–3. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say $\vec{v}_1$ and $\vec{v}_2$ are (distinct!) solutions of the equations $L\vec{x} = \vec{y}_1$ while $\vec{w}$ is a solution of $L\vec{x} = \vec{y}_2$. Answer the following in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{w}$.

a) Find some solution of $L\vec{x} = 2\vec{y}_1 - 7\vec{y}_2$.

b) Find another solution (other than $\vec{w}$) of $L\vec{x} = \vec{y}_2$.

A–4. Say you have matrices A and B.

a) If $A : \mathbb{R}^7 \to \mathbb{R}^7$, what are the possible dimensions of the kernel of A? The image of A?

b) If $B : \mathbb{R}^3 \to \mathbb{R}^5$, what are the possible dimensions of the kernel of B? The image of B?

A–5. Give an example of 2×2 matrices A and B with $AB = 0$ but $A \neq 0$ and $B \neq 0$.

PART B Four questions, 15 points each (so 60 points total).

B–1. Let $C = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. [NOTE: In this problem, there is *no partial credit* for incorrect computations.]

a) Find the inverse of C.

b) Find the inverse of C^2.

B–2. Define the linear maps A, B, and C from $\mathbb{R}^2 \to \mathbb{R}^2$ by the rules

- A rotates vectors by $\pi/2$ radians counterclockwise.
- B reflects vectors across the vertical axis.
- C orthogonal projection onto the vertical axis, so $(x_1, x_2) \to (0, x_2)$

Let M be the linear map that first applies A, then B, and finally C. Find a matrix that represents M in the standard basis for $\mathbb{R}^2$.

B–3. Let the linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ be specified by the matrix $A := \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{pmatrix}$.

a) Find a basis for the kernel of A. (CAUTION: in this problem, $\ker A \neq 0$)

b) Find a basis for the image of A.

c) With the above matrix A, is it possible to find an invertible 3×3 matrix B so that the matrix AB is invertible?

B–4. Say you are given the four data points $(-1, 0)$, $(1, 2)$, $(4, -2)$, and $(5, 3)$. Find a polynomial $p(x)$ of degree at most three that passes through these four points. [Don't bother to "simplify" your answer.]